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The Instability of a Rotating Fluid Sphere Heated From Within

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TECHNICAL REPORT NO. 8

August 15, 1954

EARTH'S MAGNETISM AND MAGNETOHYDRODYNAMICS
CONTRACT Nonr 1288(00)
OFFICE OF NAVAL RESEARCH

DEPARTMENT OF PHYSICS
UNIVERSITY OF UTAH
SALT LAKE CITY

The Instability of a Rotating Fluid Sphere Heated From Within

By Hitoshi Takeuchi* and Yasuo Shimazu

Abstract

In the present paper, the Benard's cell problem in a rotating sphere is studied. By a preliminary study in section 3, it is shown that the rotation stabilizes the convection currents and tends to make the scale of the convective cells smaller. In section 4 it is shown that each poloidal mode of the motion can, in a non-rotating sphere, be excited independently. As is shown in section 5, this is not the case in a rotating sphere. Each motion is then composed of infinite numbers of poloidal and toroidal modes. To this rather complicated case, the general results in section 3 can still be applied. In connection with the present problem, a method is given in section 8 to get an equivalent viscosity coefficient of magnetic fields.

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*) This paper (hitherto unpublished) was completed previous to Dr. Takeuchi's joining this project. It is here reproduced on account of its importance for the hydromagnetic theory of the earth's liquid core. (W. M. E.)

1. In a series of previous papers (H. Takeuchi and Y. Shimazu, 1952, 1953), it was shown that a self-exciting process is possible by which the earth's main magnetic field may be produced and maintained. The self-exciting process is considered to be maintained by the induction currents caused by the motions of the fluid of which the earth's core is composed. In order to make the study on the self-exciting dynamo complete, however, there remains another problem to be solved. That is the problem of the fluid motion itself. Is it possible that the required fluid motion takes place in the earth's core, and what are the conditions for this? Is the fluid motion appropriate to maintained the self-exciting dynamo? These questions will be considered in the present paper.

2. After considering all possibilities, it is now believed that the fluid motion in the earth's core is the convective one caused by non-homogeneous heating of the fluid (E. C. Bullard, 1949; W. M. Elsasser, 1950). In fact, our previous studies, referred to above, are based on this convection-current model. The S_2^{2c}-type velocity field in these papers is nothing but the mathematical expression for the convection current. It is this S_2^{2c}-type velocity field that makes our self-excited dynamo possible. In view of these circumstances, our immediate problem may be stated as follows:

(1) Can a stationary fluid motion of S_2^{2c}-type take place in the earth's core, and under what conditions?

(2) It must be shown that the S_2^{2c}-type velocity field is the easiest type of motion to excite.. Otherwise, the S_2^{2c}-type

velocity field may be overcome by fluid motions of other, more powerful types.

The stationary velocity field of S_2^{2c}-type reminds us of the well-known Benard cell. It may be that the fluid motion in the earth's core is of a character similar to the Benard cell in a vessel heated from below. This will be the main point of view of the present study. Many studies have been made on the convective motion of a viscous fluid heated from below. In almost all these studies, the problem is treated for plane boundaries, and the effects of rotation upon the fluid motion are not taken into account. In the next section, taking the earth's rotation into account, we shall treat the problem for plane boundaries.

3. We shall consider the fluid of density ρ, viscosity μ, kinematical viscosity $\nu = \frac{\mu}{\rho}$, thermal diffusivity k and coefficient of thermal expansion α. The fluid is contained in two walls placed horizontally at a vertical distance of h.

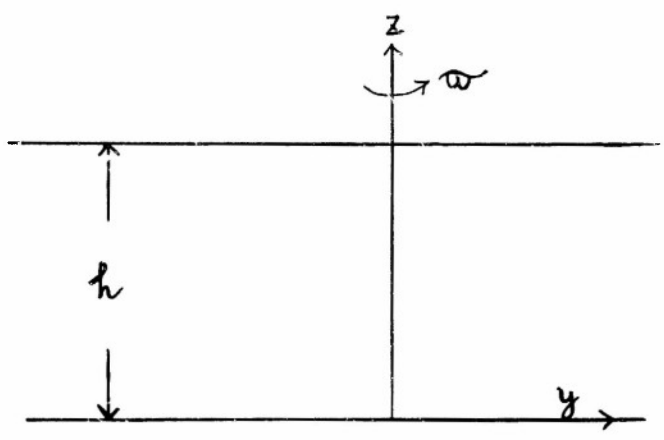

Fig. 1

The fluid is heated from below. The temperature gradient in a state of no convection is β. Referring to the rotating rectangular coordinates as shown in Fig. 1, we have the following equations

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu \nabla^2 u + 2\rho \bar{\omega} v,$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \mu \nabla^2 v - 2\rho \bar{\omega} u,$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \mu \nabla^2 w - \rho g,$$

$$\frac{DT}{Dt} = k \nabla^2 T. \tag{3.1}$$

Customary notations are used. Attaching the suffixes o to the quantities in the state of no convection, we have

$$u_o = v_o = w_o = 0 \ , \quad P_o = P_o(z) \ , \quad T_o = T_o(z),$$

$$\frac{dT_o}{dz} = -\beta z \ , \quad \rho_o = \rho_{o,o}(1 - \alpha T_o),$$

$$-\frac{\alpha P_o}{dz} - \rho_o g = 0, \tag{3.2}$$

$\rho_{o,o}$ being a certain mean density corresponding to a certain mean temperature (or height, z). We shall put

$$P = P_o + P' \ , \quad T = T_o + T' \ , \quad \rho = \rho_o + \rho' \tag{3.3}$$

for p, T and ρ in (3.1), and neglect the second order terms in u, v, w, p', T' and ρ'. Furthermore, we shall assume that the density variation of the fluid is caused by the thermal expansion only and that the fluid behaves otherwise as if it were an incompressible one. Thus we get

$$\rho = \rho_{o,o}(1-\alpha T) \ , \quad \rho' = -\rho_{o,o}\alpha T', \tag{3.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad (3.5)$$

Inserting (3.2) - (3.4) into (3.1) and denoting p', T' and $\rho_{o,o}$ as p, T and ρ anew, we have for the state of stationary convection

$$\nu\nabla^2 u - \frac{1}{\rho}\frac{\partial p}{\partial x} + 2\bar{\omega}\, v = 0,$$

$$\nu\nabla^2 v - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\bar{\omega} u = 0,$$

$$\nu\nabla^2 w - \frac{1}{\rho}\frac{\partial p}{\partial z} + \alpha\rho\, gT = 0,$$

$$k\nabla^2 T + \beta w = 0. \qquad (3.6)$$

While the above equations were obtained for an incompressible fluid, H. Jeffreys (1930) has shown that the same equations can also be applied to compressible fluids provided the density does not vary greatly within the system and provided also we interpret β as the temperature gradient in excess of the adiabatic one. Putting

$$(u, v, w, p, T) = e^{\frac{i}{h}(mu + ny)}(u(z), v(z), w(z), p(z), T(z)), \qquad (3.7)$$

we get five ordinary differential equations for five unknown functions u(z), v(z), - - - -. Putting

$$m^2 + n^2 = (\pi b)^2 , \quad z = h\,\zeta \qquad (3.8)$$

and eliminating u(z), v(z), w(z) and p(z) from (3.5) - (3.7), we have

$$\left[\left(\frac{d^2}{d\zeta^2} - \pi^2 b^2\right)^3 + \left(\frac{2\bar{\omega}h^2}{\nu}\right)^2 \frac{d^2}{d\zeta^2} + \frac{\pi^{m2} b^2 h^4 \alpha\beta g}{k\nu}\right] T = 0. \qquad (3.9)$$

In order to get a quantitative result, we shall assume

$$T = w = \frac{d^2 w}{d\zeta^2} = 0 \qquad (3.10)$$

at $\zeta = 0$ and 1. By (3.10) the temperature at the two free boundaries,

$z = 0$, and $z = h$ is kept constant. These boundary conditions are satisfied by putting

$$T \propto \sin s\pi\zeta \quad (s = 1, 2, ---). \tag{3.11}$$

Inserting (3.11) into (3.9), we get the following condition for stationary convection

$$\lambda = \frac{\pi^4}{b^2}(b^2 + s^2)^3 + \left(\frac{2\omega h^2}{\nu}\right)^2 \frac{s^2}{b^2}, \tag{3.12}$$

where

$$\lambda = \frac{h^4\alpha\beta g}{k\nu}. \tag{3.13}$$

In the case where the value of λ in (3.13) is larger (or smaller) than that given by the right-hand side of (3.12), the fluid motion is considered to be unstable (or stable) and growing (or damped). Keeping $\left(\frac{2\bar{\omega}h^2}{\nu}\right)^2$ as a parametric constant and making use of (3.12), we can determine the critical value λ as a function of b and s. The value of λ thus considered may become a minimum for certain values of b and s. As the value of λ in (3.12) is a monotonically increasing function of s, we have $s = 1$ as the value of s for which λ in (3.12) is a minimum. Thus, putting $s = 1$ ans $\frac{d\lambda}{db} = 0$ in (3.12), we can determine the value of b for which λ has its minimum. The value obtained is an increasing function of $\left(\frac{2\bar{\omega}h^2}{\nu}\right)^2$. In any case, inserting the value of b into (3.12), we get the minimum value of λ. This minimum value is once more an increasing function of $\left(\frac{2\bar{\omega}h^2}{\nu}\right)^2$. The reason for the existence of these relations is as follows: In a convective fluid motion the temperature gradient is a destabilizing factor, while viscosity and Coriolis force are stabilizing factors. The stabilizing action of the Coriolis force is studied and appreciated in dynamical meteorology. In (3.12), the second term of the right-hand side of the equation denotes this stabilizing action of the Coriolis force. The first term of the right-hand side denotes also the stabilizing action of the viscosity, while the left-hand side of (3.12) shows the destabilizing action of inhomogeneous heating. Equation (3.12) shows that the convective fluid motion can exist stationarily when these stabilizing and unstabilizing actions balance each other. Thus we can understand the λ (minimum) $\sim (\frac{2\bar{\omega}h^2}{\nu})^2$ relation obtained above. Furthermore, as is shown in dynamical meteorology, the stabilizing action of the Coriolis force is more powerful for larger-scale fluid motions, while that of the viscosity is more powerful for smaller-scale motions. These circumstances are reflected in (3.12). Thus, the first (second) term of the right-hand side of (3.12) becomes larger for larger (smaller) values of b, that is, for smaller (larger) scale motion. In short, the action of the Coriolis force makes fluid motions of a smaller scale easier to excite relative to larger-scale motions. This is the reason for the $b \sim (\frac{2\bar{\omega}h^2}{\nu})^2$ relation obtained above. From the above physical considerations, we may safely assume that the similar relations,, namely, $\lambda(\min) \sim (\frac{2\bar{\omega}h^2}{\nu})^2$ and $b \sim (\frac{2\bar{\omega}h^2}{\nu})^2$ will also be obtained under boundary conditions other than those assumed here. The existence of the relation $b \sim (\frac{2\bar{\omega}h^2}{\nu})^2$ is favorable for our present purpose. The reason is as follows: Since the S_2^{2c}-type fluid motion is of a smaller scale than some of the more symmetrical motions, we cannot expect that the S_2^{2c}-type motion is the easiest motion to be excited (as was expected in (2) in § 2) without the relation $b \sim (\frac{2\bar{\omega}h^2}{\nu})^2$. In short, while the S_2^{2c} motion is not the easiest motion to be excited in the case when $\bar{\omega} = 0$, we may expect it will become so for a certain value of $(\frac{2\bar{\omega}h^2}{\nu})^2$. This is the most important result obtained in this section.

4. We shall now consider convective fluid motions in a spherical vessel of radius a. Referring to the rotating coordinates in Fig. 2, we have the following equations:

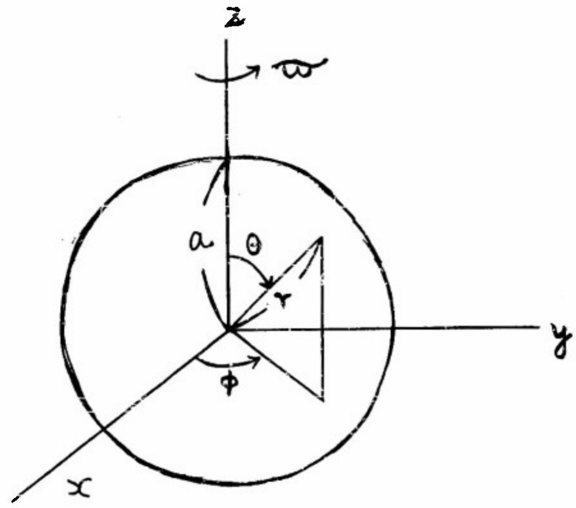

Fig. 2

$$\rho \frac{D\vec{u}}{Dt} = - \operatorname{grad} p + \mu \nabla^2 \vec{u} - 2\rho(\vec{\omega} \times \vec{u}) - \rho \vec{g},$$

$$\frac{DT}{Dt} = k\nabla^2 T + H \qquad ,(4.1)$$

where $\vec{u}$, $\vec{\omega}$ and $\vec{g}$ denote the displacement (of the fluid relative to the above coordinates), rotation (of the spherical vessel) and gravity vector respectively. A source of heat is assumed. The rate of heat generation is assumed such that, in the absence of heat conduction and convection, the temperature would rise at a uniform rate H. Thus in the state of no convection, we have

$$0 = \frac{\partial T_o}{\partial t} = k\nabla^2 T_o + H \qquad .(4.2)$$

In a sphere of radius a, this gives

$$T_o = \frac{Ⓗ}{6k}(a^2 - r^2) = \frac{\beta}{2a}(a^2 - r^2) \qquad ,(4.3)$$

where

$$\beta = \frac{Ⓗ a}{3k} = \frac{Qa}{3\rho ck} \qquad (4.4)$$

is the temperature gradient at $r = a$. In (4.4), Q and ρck are the heat generation (per unit time and volume) and heat conductiviyt, respectively. In the same way as in § 3, we get from (4.1) - (4.4)

$$\nu \nabla^2 \vec{u} - \frac{1}{\rho} \operatorname{grad} p - 2(\vec{\bar{\omega}} \times \vec{u}) + \alpha \vec{g} T = 0 \qquad ,(4.5)$$

$$\nabla^2 T = \frac{u_r}{k} \frac{dT_o}{dr} = -\frac{\beta}{ka} r u_r \qquad ,(4.6)$$

$$\operatorname{div} \vec{u} = 0 \qquad .(4.7)$$

In (4.6), u_r means the radial velocity in the spherical coordinates in Fib. 2. In these spherical coordinates, we have also

$$\vec{u} = (u_r, u_\theta, u_\phi), \quad \vec{\bar{\omega}} = (\bar{\omega} \cos\theta, -\bar{\omega} \sin\theta, 0),$$

$$\vec{g} = (g, 0, 0) \qquad .(4.8)$$

The forms of velocity vectors $\vec{u}$ satisfying (4.7) are known (H. Takeuchi, 1950):

<u>Type I</u>

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = F_{n,m}(r) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} \end{pmatrix} \bar{\omega}_n + G_{n,m}(r) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \bar{\omega}_{n,m},$$

$$r u_r = (n F_{n,m} + r^2 G_{n,m}) \bar{\omega}_{n,m},$$

$$\begin{pmatrix} r u_\theta \\ r u_\phi \end{pmatrix} = F_{n,m} \begin{pmatrix} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\sin\theta \, \partial\phi} \end{pmatrix} \bar{\omega}_{n,m} \qquad ,(4.9)$$

where $\overline{\omega}_n$ is a solid spherical harmonic of degree n and is expressed as follows

$$\overline{\omega}_{n,m} = r^n s_{n,m}(\theta, \phi) = r^n P_n^m(\theta) e^{im\phi} \qquad .(4.10)$$

In (4.10), $s_{n,m}$ is a surface spherical harmonic, and P_n^m is a Legendre function. For the velocity vector of type 1, we have also

$$\operatorname{div} \vec{u} = 0 = \left[\frac{n}{r} \frac{dF_{n,m}}{dr} + r \frac{dG_{n,m}}{dr} + (n+3) G_{n,m} \right] \overline{\omega}_{n,m} \qquad ,(4.11)$$

$$\begin{matrix} (\nabla^2 \vec{u})_x \\ (\nabla^2 \vec{u})_y \\ (\nabla^2 \vec{u})_z \end{matrix} = f_{n,m}(r) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \overline{\omega}_{n,m} + g_{n,m} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \overline{\omega}_{n,m}$$

$$\cancel{(\nabla^2 \vec{u})_z}$$

$$r(\nabla^2 \vec{u})_r = (n f_{n,m} + r^2 g_{n,m})\, \overline{\omega}_{n,m}$$

$$\begin{matrix} r(\nabla^2 \vec{u})_\theta \\ r(\nabla^2 \vec{u})_\phi \end{matrix} \begin{matrix} = f_{n,m} \\ = \cancel{f_{n,m}} \end{matrix} \begin{pmatrix} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\sin\theta\, \partial\phi} \end{pmatrix} \begin{matrix} \overline{\omega}_{n,m} \\ \cancel{\overline{\omega}_{n,m}} \end{matrix} \qquad ,(4.12)$$

in which

$$f_{n,m} = \frac{d^2 F_{n,m}}{dr^2} + \frac{2n}{r} \frac{dF_{n,m}}{dr} + 2\, G_{n,m}$$

$$g_{n,m} = \frac{d^2 G_{n,m}}{dr^2} + \frac{2(n+2)}{r} \frac{dG_{n,m}}{dr} \qquad .(4.13)$$

Another type of $\vec{u}$ satisfying (4.7) is as follows:

<u>Type II</u>

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = L_{n,m}(r) \begin{pmatrix} z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \\ x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \\ y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{pmatrix} \overline{\omega}_{n,m}$$

$$u_r = 0,$$

$$u_\theta = L_{n,m} \frac{\partial \bar{\omega}_{n,m}}{\sin \partial\partial\phi},$$

$$u_\phi = - L_{n,m} \frac{\partial \bar{\omega}_{n,m}}{\partial\theta} \qquad ,(4.14$$

$$\operatorname{div} \vec{u} = 0 \qquad ,(4.15)$$

$$\begin{matrix} (\nabla^2 u)_x \\ (\nabla^2 u)_y \\ (\nabla^2 u)_z \end{matrix} = l_{n,m}(r) \begin{pmatrix} y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \end{pmatrix} \bar{\omega}_{n,m}$$

$$(\nabla^2 u)_r = 0$$

$$(\nabla^2 u)_\theta = l_{n.m}(r) \frac{\partial \bar{\omega}_{n,m}}{\sin \theta\partial\phi}$$

$$(\nabla^2 u) = I_{n.m} \frac{\partial \bar{\omega}_{n.m}}{\partial\theta} \qquad ,(4.16)$$

where

$$I_{n.m} = \frac{d^2 L_{n.m}}{dr^2} + \frac{2(n+1)}{r} \frac{dL_{n.m}}{dr} \qquad .(4.17)$$

Taking (4.5) and (4.6) into consideration, we may assume

$$T = T_{n.m}(r)\bar{\omega}_{n.m}, \quad P = \pi_{n.m}(r)\bar{\omega}_{n.m} \qquad (4.18)$$

for u of type I. For u of type II, we may take

$$T = 0, \quad P = 0 \qquad .(4.19)$$

The Laplacian of T in (4.18) is as follows

$$\nabla^2 T = t_{n.m}\bar{\omega}_{n.m}, \quad t_{n.m} = \frac{d^2 T_{n.m}}{dr^2} + \frac{2(n+1)}{r} \frac{dT_{n.m}}{dr} \qquad .(4.20)$$

Accordingly equation (4.6) is satisfied by putting

$$t_{n.m} = -\frac{\beta}{ka}(nF_{n.m} + r^2 G_{n.m}) \quad .(4.21)$$

Furthermore, if we omit the term $2(\vec{\omega} \times \vec{u})$ in (4.5), we may satisfy equation (4.5) by putting

$$\nu(nf_{n.m} + r^2 g_{n.m}) - \frac{1}{\rho r^{n-1}} \frac{d(r^n \pi_{n.m})}{dr} + \alpha g r T_{n.m} = 0, (4.22)$$

$$\nu f_{n.m} - \frac{1}{\rho}\pi_{n.m} = 0 \quad ,(4.23)$$

$$l_{n.m} = 0 \text{ or } L_{n.m} = 0 \quad .(4.24)$$

Thus, when $\bar{\omega} = 0$, we get (4.11) and (4.21) - (4.24) from (4.5) - (4.7). These are five relations among five unknowns $F_{n.m}$, $G_{n.m}$, $L_{n.m}$, $T_{n.m}$, and $\pi_{n.m}$. Vectors of type I and II are usually called poloidal and toroidal, respectively. Thus, as is seen from (4.24) when $\bar{\omega} = 0$, we can satisfy (4.5) - (4.7) by a single poloidal u and the corresponding T and P. This will not be the case when $\bar{\omega} \neq 0$. The existence of the term $2(\vec{\omega} \times \vec{u})$ makes the latter problem more complicated. A method of dealing with this complication will be given in the next section.

5. We denote $\vec{u}$ in (4.9) and (4.14) by $\vec{u}_{n.m}(I)$ and $\vec{u}_{n.m}(II)$, respectively, and shall try to satisfy (4.5) - (4.7) by putting

$$\vec{u} = \sum_n \vec{u}_{n.m}(I) + \vec{u}_{n.m}(II) \quad ..(5.1)$$

The ϕ parts of all $\vec{u}_{n.m}(I)$ and $\vec{u}_{n.m}(II)$ may be taken to be cos mϕ (m = 0, 1, 2, . . .). The reason why we may assume these forms can be understood from (4.5) - (4.20). Equation (4.11) and (4.21 still hold for each pair of ($F_{n.m}$, $G_{n.m}$, $T_{n.m}$) thus introduced. We now make the following transformations of variables

$$\xi = \frac{r}{a}, \quad F_{n.m}(\xi) = \frac{1}{a^2} F_{n.m}(r), \quad G_{n.m}(\xi) = G_{n.m}(r),$$

$$L_{n.m}(\xi) = L_{n.m}(r), \quad T_{n.m}(\xi) = T_{n.m}(r), \quad \pi_{n.m}(\xi) = \pi_{n.m}(r). \quad (5.2)$$

We shall denote $\dfrac{d^2F_{n.m}(\xi)}{d\xi^2} + \dfrac{2n}{\xi}\dfrac{dF_{n.m}}{d\xi} + 2G_{n.m}(\xi)$, - - - by $f_{n.m}(\xi)$, - - -. By using these notations, (4.11) and (4.21) can be rewritten as follows

$$t_{n.m}(\xi) = -\frac{\beta a^3}{k}\left[nF_{n.m}(\xi) + \xi^2 G_{n.m}(\xi)\right] \qquad (5.3)$$

$$\left[nf_{n.m}(\xi) + \xi^2 g_{n.m}(\xi)\right] - \frac{1}{\rho\nu\xi^{n-1}}\frac{d(\xi^n \pi_{n.m})}{d\xi} + \frac{\alpha g a}{\nu}\xi\, T_{n.m}(\xi) = 0 \quad (5.4)$$

In order to get more relations among unknown functions, operating with $\vec{u}_{n.m}(I)$ (which will be denoted by $\vec{u}_\beta(I)$ hereafter) upon (4.5) and integrating, we get

$$\int_0^a \int_0^\pi \int_0^{2\pi} (4.5) \cdot \vec{u}_\beta(I)\, r^2 \sin\theta\, dr d\theta d\phi = 0 \qquad (5.5)$$

Inserting (5.1) and (5.2) into (4.5) in (5.5) and carrying out the integration, we have

$$\int_0^1 C(\xi)\,\xi^{2n\beta}\,(n_\beta F_\beta + \xi^2 G_\beta)\,d\xi$$

$$+ \int_0^1 D(\xi)\,\xi^{2n\beta}\,F_\beta\,d\xi = 0 \qquad ,(5.6)$$

or

$$C(\xi) = (n_\beta f_\beta + \xi^2 g_\beta) - \frac{1}{\rho\nu\xi^{n_\beta - 1}}\frac{d(\xi^{n_\beta}\pi_\beta)}{d\xi} + \frac{\alpha g a}{\nu}\xi\, T_\beta$$

$$-\frac{2\bar{\omega}a^2}{\nu}\left[I_0(\beta)\right]^{-1}\sum_\alpha \xi^{n_\alpha - n_\beta + 1}\, I_1(\beta\cdot\alpha) L_\alpha = 0 \qquad ,(5.7)$$

$$D(\xi) = f_\beta - \frac{\pi_\beta}{\rho\nu} - \frac{2\bar{\omega}a^2}{\nu}\left[n_\beta(n_\beta+1)\, I_0(\beta)\right]^{-1}$$

$$\sum_{\alpha} \xi^{n_\alpha - n_\beta + 1} I_2(\alpha\cdot\beta)\, L_\alpha = 0 \qquad ,(5.8)$$

where

$$I_o(\beta) = \int_o^\pi (P_\beta)^2 \sin\theta\, d\theta \qquad ,(5.9)$$

$$I_1(\alpha\cdot\beta) = \int_o^\pi P_\alpha \frac{dP_\beta}{d\theta} \sin^2\theta\, d\theta \qquad ,(5.10)$$

$$I_2(\alpha\cdot\beta) = I_2(\beta\cdot\alpha) = \int_o^\pi \left(\frac{m^2}{\sin^2\theta} P_\alpha P_\beta + \frac{dP_\alpha}{d\theta}\frac{dP_\beta}{d\theta}\right)\sin\theta\cos\theta\, d\theta \qquad ,(5.11)$$

$$P_\alpha = P_{n_\alpha}^{m_\alpha}(\theta), \quad F_\beta = F_{n_\beta, m_\beta}, \; - - - \qquad ,(5.12)$$

Similarly, operating with $u_\beta(II)$ upon (4.5) and proceeding in the same way as above, we get

$$\int_o^1 E(\xi)\, \xi^{2(n_\beta+1)} L_\beta\, d\xi = 0 \qquad ,(5.13)$$

or

$$F(\xi) = 1_\beta + \frac{2\bar{\omega}a^2}{\nu}\left[n_\beta(n_\beta+1)\, I_o(\beta)\right]^{-1} \sum_\alpha \xi^{n_\alpha - n_\beta - 1}$$

$$\left[F_\alpha I_1(\alpha\cdot\beta) + (n_\alpha F_\alpha + \xi^2 G_\alpha)\, I_2(\alpha\cdot\beta)\right] \qquad .(5.14)$$

If equations (5.3), (5.4), (5.7), (5.8) and (5.14) are written down for all β's concerned, we see that we have sufficient numbers of equations to determine the unknown radial functions. As was said immediately after (6.1), the values of m_β of these radial functions are constants equal to a fixed value m, say. Furthermore, as is seen from (5.7), (5.8) and (5.14), the velocity vectors $\vec{u}_\alpha$ and $\vec{u}_\beta$ with $I_1(\alpha,\beta)$, $I_2(\alpha,\beta) \neq 0$ may be considered to be coupled by the action of Coriolis force. In view of these circumstances, we may call $I_1(\alpha,\beta)$ and $I_2(\alpha,\beta)$ the coupling integrals. Now it can be easily shown that

$$I_0(\beta) = \frac{2}{2n_\beta+1}\frac{(n_\beta+m_\beta)!}{(n_\beta-m_\beta)!} \quad ,(5.15)$$

$$I_2(\alpha,\beta) = I_1(\alpha,\beta) + n_\beta(n_\beta+1)\int_0^\pi P_\alpha P_\beta \sin\theta\cos\theta\, d\theta \quad (5.16)$$

and that the coupling integrals $I_1(\alpha,\beta)$ and $I_2(\alpha,\beta)$ vanish unless $n_\alpha - n_\beta = \pm 1$. The values of $I_1(\alpha,\beta)$ and $I_2(\alpha,\beta)$ for several α and β are shown in Table I. Having thus established the fundamental equations, our next task is to consider the boundary conditions. By the assumption of a "viscous" fluid, we have

$$\vec{u}_{n.m}(I),\ \vec{u}_{n.m}(II) = 0 \text{ at } \xi = 1 \quad .(5.17)$$

As the thermal boundary condition we shall take tentatively

$$T = 0 \text{ at } \xi = 1 \quad .(5.18)$$

By (5.18) is meant that the temperature at $r = a$ is kept constant. This is a plausible condition for the boundary of the earth's core. Taking (4.9), (4.14) and (4.18) into consideration, we have

$$nF_{n,m} + \xi^2 G_{n,m} = G_{n,m} = L_{n,m} = T_{n,m} = 0 \text{ at } \xi = 1 \quad .(5.19)$$

$F_{n,m}$, $G_{n,m}$, - - - must also be finite at $r = 0$ (or $\xi = 0$). Thus our problem is reduced to solving (5.3), (5.4), (5.7), (5.8) and (5.14) under the boundary condtions (5.19). It is easily seen that this is an eigen-value problem for $\lambda = \frac{\alpha\beta g_0 a^4}{k\nu}$ which depends on the parameter $\frac{\bar{\omega}a^2}{\nu}$.

6. By using the results of the last section, we find sets of fluid motions as shown below

$$u_{1.0}(I) \rightleftarrows u_{2.0}(II) \rightleftarrows u_{3.0}(I) \text{ - - - } \quad ,(6.1)$$

$$u_{1.0}(II) \rightleftarrows u_{2.0}(I) \rightleftarrows u_{3.0}(II) \text{ - - - } \quad ,(6.2)$$

$$u_{1.1}(I) \rightleftarrows u_{2.1}(II) \rightleftarrows u_{3.1}(I) \text{ - - - } \quad ,(6.3)$$

$$u_{1.1}(II) \rightleftarrows u_{2.1}(I) \rightleftarrows u_{3.1}(II) \; - - - \qquad ,(6.4)$$

$$u_{2.2}(I) \rightleftarrows u_{3.2}(II) \rightleftarrows u_{4.2}(I) \; - - - \qquad ,(6.5)$$

$$u_{2.2}(II) \rightleftarrows u_{3.2}(I) \rightleftarrows u_{4.2}(II) \; - - - \qquad ,(6.6)$$

$$u_{3.3}(I) \rightleftarrows u_{4.3}(II) \rightleftarrows u_{5.3}(I) \; - - - \qquad ,(6.7)$$

etc.

Generally speaking, there exist sets of fluid motions which contain $u_{n.m}$(I, or II) with $n = m$ as their first member. As is easily understood, the case when $n = m = 0$ is an exceptional one. This is the most important result obtained in the last section. $\vec{u}_{2.2}(I)$ in (6.5) is usually denoted by S_2^2 and it is this S_2^2-type velocity which was referred to in section 2. In a rotating sphere, the S_2^2-type motion cannot exist by itself. It must be accompanied by the fluid motions of $\vec{u}_{3.2}(II)$, $\vec{u}_{4.2}(I)$, - - - types. We shall call these latter quasi-S_2^2-type motions. For the quasi-S_2^2-type motion, we have from (5.3) and (5.4)

$$\frac{2}{\xi}\frac{dF_2}{d\xi} + \xi\frac{dG_2}{d\xi} + 5G_2 = 0 \qquad ,(a)$$

$$\frac{4}{\xi}\frac{dF_4}{d\xi} + \xi\frac{dG_4}{d\xi} + 7G_4 = 0 \qquad ,(b) \text{ etc.} \qquad (6.8)$$

$$\frac{d^2T_2}{d\xi^2} + \frac{6}{\xi}\frac{dT_2}{d\xi} = -\frac{\beta a^3}{k}(2F_2+\xi^2G_2) \qquad ,(a)$$

$$\frac{d^2T_4}{d\xi^2} + \frac{10}{\xi}\frac{dT_4}{d\xi} = -\frac{\beta a^3}{k}(4F_4+\xi^2G_4) \qquad ,(b) \text{ etc.} \qquad (6.9)$$

In (6.8) and (6.9), we write simply F_2, G_2 - - - for $F_{2.2}(\xi)$, $G_{2.2}(\xi)$, - - -. Corresponding to $C(\xi) = 0$ in (5.7) we have

$$\Big[\,2\Big(\frac{d^2F_2}{d\xi} + \frac{4}{\xi}\frac{dF_2}{d\xi} + 2G_2\Big) + \xi^2\Big(\frac{d^2G_2}{d\xi^2} + \frac{8}{\xi}\frac{dG_2}{d\xi} - \frac{1}{\rho\nu\xi}\frac{d(\xi^2\pi_2)}{d\xi}$$

$$+\frac{\alpha g_o a}{\nu}\xi^2 T_2 + \frac{5}{24}\Omega\left(\frac{192}{7}\right)\xi^2 L_3 = 0 \qquad ,(a)$$

$$\left[4\left(\frac{d^2F_4}{d\xi^2} + \frac{8}{\xi}\frac{dF_4}{d\xi} + 2G_4\right) + \xi^2\left(\frac{d^2G_4}{d\xi^2} + \frac{12}{\xi}\frac{dG_4}{d\xi}\right) - \frac{1}{\rho\nu\xi^3}\frac{d(\xi^4\pi_4)}{d\xi}\right.$$

$$+\frac{\alpha g_o a}{\nu}\xi^2 T_4 - \frac{1}{40}\Omega\left(\frac{480}{7}L_3 - \frac{480\text{x}7}{11}\xi^2 L_5\right) = 0 \qquad ,(b) \text{ etc.} \qquad (6.10)$$

In (6.10) $\quad \Omega = \dfrac{\omega a^2}{\nu} \qquad (6.11)$

and g is put equal to

$$g = g_o\xi \qquad ,(6.12)$$

where g_o is the gravity at $\xi = 1$, that is, at the surface of the earth's core. The gravity g in a uniform self-gravitating sphere may be shown to vary as in (6.12). Next, corresponding to $D(\xi) = 0$ in (5.8), we have

$$\frac{d^2F_2}{d\xi^2} + \frac{4}{\xi}\frac{dF_2}{d\xi} + 2G_2 - \frac{\pi_2}{\rho\nu} - \frac{5}{144}\Omega\frac{384}{7}\xi^2 L_3 = 0 \qquad ,(a)$$

$$\frac{d^2F_4}{d\xi^2} + \frac{8}{\xi}\frac{dF_4}{d\xi} + 2G_4 - \frac{\pi_4}{\rho\nu} - \frac{\Omega}{800}\left(\frac{480\text{x}5}{7}L_3 + \frac{480\text{x}28}{11}\xi^2 L_5\right) = 0 \qquad ,(b) \text{ etc.} \qquad (6.13)$$

Lastly, corresponding to $E(\xi) = 0$ in (5.14) we have

$$\frac{d^2L_3}{d\xi^2} + \frac{8}{\xi}\frac{dL_3}{d\xi} + \frac{7}{1440}\Omega\left\{\xi^{-2}\left[-\frac{192}{7}F_2 + \frac{384}{7}(2F_2 + \xi^2 G_2)\right]\right.$$

$$\left. + \left[\frac{480}{7}F_4 + \frac{480\text{x}5}{7}(4F_4 + \xi^2 G_4)\right]\right\} = 0 \qquad ,(a)$$

$$\frac{d^2L_5}{d\xi^2} + \frac{12}{\xi}\frac{dL_5}{d\xi} + \frac{11}{210\text{x}120}\Omega\left\{\xi^{-2}\left[-\frac{480\text{x}7}{11}F_4 + \frac{480\text{x}28}{11}(4F_4\right.\right.$$

$$\left.\left. + \xi^2 G_4)\right] + \left[\frac{2800\text{x}24}{11\text{x}13}F_6 + \frac{2800\text{x}168}{11\text{x}13}(6F_6 + \xi^2 G_6)\right]\right\} = 0 \qquad ,(b) \text{ etc.} \qquad (6.14)$$

If we omit $\vec{u}_{n.2}$(I, or II) with $n \geq 4$ in (6.5), we get the results (a) in (6.8) - (6.14). In that case, the F_4 and G_4 terms in (6.14) (a) must of course be omitted. The results thus obtained will be called the approximation (a). Approximations (b) and (c) will be defined similarly. In the approximation (a) we have five unknowns, F_2, G_2, π_2, T_2, L_3. In the approximation (b) the number of unknowns becomes 10.

Proceeding in this way to approximations (c) and (d)--- we shall obtain the required solution for (6.5). Boundary conditions to be satisfied are shown in (5.19). Although the above results are obtained for the quasi-S_2^2-type motion, similar results may be obtained for (6.1) - (6.7). It is, however, almost hopeless to solve the eigen-value problem by trial-and-error methods. A method of dealing with the difficulties will be given in the next section.

7. Taking (5.19) into consideration, we shall put

$$\left.\begin{aligned} nF_{n.m} + \xi^2 G_{n.m} &= (1-\xi)^2 \, (A_o + A_1 \xi + - - -), \\ L_{n.m} &= (1-\xi) \, (B_o + B_1 \xi + - - -) \end{aligned}\right\} \qquad ,(7.1)$$

where A_o, B_o, A_1, B_1 - - - are undetermined constants. Transforming (5.3) into

$$(n+1)G_{n.m} = - \frac{d(nF_{n.m} + \xi^2 G_{n.m})}{\xi d\xi} \qquad ,(7.2)$$

we can calculate the value of $G_{n.m}$ by (7.1) and (7.2). Inserting $G_{n.m}$ thus obtained into (7.1), we get $F_{n.m}$. The form of $nF_{n.m} + \xi^2 G_{n.m}$ in (7.1) is so adjusted as to make $F_{n.m}$ vanish at $\xi = 1$. Next, inserting (7.1) into (5.4), we have the following differential equation for $T_{n.m}$

$$\frac{d^2T_{n.m}}{d\xi^2} + \frac{2(n+1)}{\xi}\frac{dT_{n.m}}{d\xi} = -\frac{\beta a^3}{k}\sum_{\bar{j}} A'_{\bar{j}}\xi^{\bar{j}} \quad ,(7.3)$$

where A'_j is a certain constant which is a function of A_o, A_1 - - - in (7.1). The equation (7.3) is solved in the form

$$T_{n.m}\left(\frac{\beta a^3}{k}\right)^{-1} = A'' - \sum_{\bar{j}} \frac{A'_j}{(j+2)(j+3+2n)}\xi^{\bar{j}+2} \quad ,(7.4)$$

where A'' is a constant which is determined so as to make $T_{n.m}$ in (7.4) satisfy the condition (5.19). The value of A'' thus adjusted is

$$A'' = \sum_{\bar{j}} \frac{A'_j}{(j+2)(j+3+2n)} \quad (7.5)$$

Inserting (7.1) and (7.2) into (5.8), we can calculate $\pi_{n.m}$. Thus, with $nF_{n.m} + \xi^2 G_{n.m}$ and $L_{n.m}$ in (7.1), we can satisfy three groups of differential equations, (5.3), (5.4) and (5.8). There remain, however, two groups of differential equations to be solved. They are (5.7) and (5.14). We have also two groups of undetermined constants A_o, A_1, - - - B_o, B_1 - - -. We shall now try to satisfy (5.7) and (5.14) approximately by choosing these constants adequately. Inserting (7.1) - (7.5) into (5.13) and

$$\int_0^1 C(\xi)\xi^{2n_\beta}(n_\beta F_\beta + \xi^2 G_\beta)d\xi = 0 \quad ,(7.6)$$

(see (5.6) and the equation $D(\xi) = 0$ above satisfied) we have equations of the following forms

$$A_o \int_0^1 C(\xi)\xi^{2n_\beta}(1-\xi)^2 d\xi = 0,$$

$$A_1 \int_0^1 C(\xi)\xi^{2n_\beta+1}(1-\xi)^2 d\xi = 0, \text{ - - -} \quad ,(7.7)$$

$$B_o \int_0^1 E(\xi)\xi^{2(n_\beta+1)}(1-\xi)d\xi = 0,$$

$$B_1 \int_0^1 E(\xi)\xi^{2(n_\beta+1)+1}(1-\xi)d\xi = 0, \; - - - \qquad (7.8)$$

These equations will be used to determine the values of A_o, A_1, ---, B_o, B_1, - - -. In order to show the way of determining A_o, - - -, we shall take up the approximation (a) for the case (6.5). We shall, furthermore, take only the first terms of $2F_{2.2} + \xi^2 G_{2.2}$ and $L_{2.2}$ in (7.1). The equations that determine A_o and B_o are obtained by (7.7) and (7.8), as follows:

$$\left(\frac{571}{11.\ 18.\ 30.\ 21.\ 28.\ 13}\lambda - \frac{2}{15}\right)A_o + \left(\frac{2\Omega}{21.\ 9}\right)B_o = 0,$$

$$\left(\frac{-\Omega}{6.\ 45.\ 15}\right)A_o + \left(-\frac{1}{9}\right)B_o = 0 \qquad (7.9)$$

In order that these equations be compatible, the determinant formed by the coefficients of A_o and B_o must be equal to zero. The equation for $\lambda = \dfrac{a^4\alpha\beta g_o}{k\nu}$ thus obtained is

$$\lambda = 10602.5 + 1.86994\Omega^2, \quad \Omega = \frac{\overline{\omega}a^2}{\nu} \qquad (7.10)$$

It is to be noted that this is a relation of the same type as (3.12). The values of λ in (7.10) have been calculated for several Ω^2's and are shown in the column (a.1) in Table II. The results obtained by taking the first two terms of $2F_{2.2} + \xi^2 G_{2.2}$ and $L_{2.2}$ are shown in the column (a.2) in the same table. Similarly, the (b.1) and (c.1) values of λ are shown in Table II. From the results in this table we see that we may use the (c.1) values of λ for the exact λ in the case (6.5). The (c.1) values of λ for the cases (6.1) - (6.7) are calculated and are shown in Table III. A discussion of the results given in Table III will appear in the next section.

8. As is seen from Table III, if we put $\overline{\omega} = 0$ in (6.1), (6.2), - - -, we get the eigen-values $\lambda = 0.811 \times 10^4$, 1.06×10^4, - - -.

The eigen-functions corresponding to these eigen-values are of pure S_1^o, S_2^o, S_1^1, S_2^1, S_2^2, S_3^2, and S_3^3 types. These are nothing but the results obtained by H. Jeffreys, M. E. M. Bland and S. Chandrasekhar (H. Jeffreys and M. E. M. Bland, 1951: S. Chandrasekhar, 1952). The minimum values of λ in the cases $\Omega^2 = 0, 10^2, 10^3$ and 10^4 are given by the fluid motions of the quasi-S_1^o, S_1^1, S_1^1, and S_2^2 types, respectively. The minimum λ increases with Ω^2. These relations correspond to the $b \sim (\frac{2\bar{\omega}h^2}{\nu})^2$ and $\lambda(\min) \sim (\frac{2\bar{\omega}h^2}{\nu})$ relations of § 3. From our present point of view, it is important that the quasi-S_2^2-type motion becomes the easiest type to be excited when $\Omega^2 = 10^4$. We shall now estimate the value of the kinematical viscosity ν corresponding to

$$\Omega^2 \doteqdot 10^4, \quad \Omega \doteqdot \frac{\bar{\omega}a^2}{\nu} = 10^2 \qquad .(8.1)$$

Putting $\bar{\omega} = \frac{2\pi}{86400}$ (sec)$^{-1}$ and $a = 3.4 \times 10^8$ cm into (8.1), we have $\nu \doteqdot 10^{11}$ cm^2/sec. In short, if the kinematical viscosity of the fluid in the earth's core is about 10^{11} cm^2/sec, we shall have the fluid motion of the quasi-S_2^2-type. This value of ν is very large compared with that which was hitherto estimated for the earth's core, i.e., $\nu \doteqdot 10^{-2} \sim 10^{-3}$ cm^2/sec (E. C. Bullard, 1949). This contradiction may be avoided as follows. While the viscosity $\nu = 10^{-2} \sim 10^{-3}$ above referred to is the molecular one, what our present study is concerned with may be some kind of equivalent viscosity. In our present study, for example, the existence of a magnetic field in the earth's core and its prohibitive action on the convection (S. Chandrasekhar, 1952) have not been taken into account. The equivalent viscosity of the magnetic field may be estimated as follows: The equations of motion in a magnetic field are

$$\frac{\partial \vec{u}}{\partial t} = \frac{1}{4\pi\rho} \, (\mathrm{curl}\ \vec{H} \times \vec{H}) + \nu \nabla^2 u + - - - \qquad ,(8.2)$$

The assumption

$$\frac{1}{4\pi\rho}(\text{curl } \vec{H} \times \vec{H}) \sim \nu \nabla^2 u \quad ,(8.3)$$

may be used to obtain an expression for the equivalent viscosity of the magnetic field,

$$\nu = \frac{LH^2}{4\pi\rho U} \quad .(8.4)$$

In (8.4), L, H and U are representative values of length, magnetic field and velocity. Inserting $\rho = 10$, $L = 3 \times 10^8$, $H = 40$ and $U = 0.03$ into (8.4), we get $\nu \sim 1.6 \times 10^{11}$ cm^2/sec which is of the order of magnitude required in (8.1). We shall now consider the corresponding value of

$$\lambda = 2.3 \times 10^4 \quad (8.5)$$

obtained in the last section. Inserting $g_0 = 10^3$, $a = 3.4 \times 10^8$, $\alpha = 10^{-5}$, $k = 10^{-1}$ and $\nu = 10^{11}$ into (8.5), we get

$$\beta = 2 \times 10^{-18} \frac{°C}{cm} \quad (8.6)$$

As was stated in section 3, β is the temperature gradient (at the surface of the earth's core) in excess of the adiabatic one. The latter is estimated in the core to be 10^{-5}. Thus the quasi-S_2^2-motion can exist in a stationary state with a thermal gradient which differs but little from the adiabatic one. In estimating β in (8.6), we used the value of the molecular thermal diffusivity for k. Even if we substitute some kind of eddy diffusivity for it, the result obtained above will not be changed much.

Table I.

$n_\alpha^{\ m}$	$n_\beta^{\ m}$	I_1	I_2
1^0	2^0	$-\frac{4}{5}$	$\frac{4}{5}$
3^0	2^0	$\frac{12}{35}$	$\frac{48}{35}$
3^0	4^0	$-\frac{40}{63}$	$\frac{40}{21}$
5^0	4^0	$\frac{40}{99}$	$\frac{80}{33}$
5^0	6^0	$-\frac{84}{11\times13}$	$\frac{15\times28}{11\times13}$
1^1	2^1	$-\frac{12}{5}$	$\frac{12}{5}$
3^1	2^1	$\frac{96}{35}$	$\frac{384}{35}$
3^1	4^1	$-\frac{200}{21}$	$\frac{200}{7}$
5^1	4^1	$\frac{320}{33}$	$\frac{640}{11}$
5^1	6^1	$-\frac{735\times4}{11\times13}$	$\frac{735\times20}{11\times13}$
2^2	3^2	$-\frac{192}{7}$	$\frac{384}{7}$
4^2	3^2	$\frac{480}{7}$	$\frac{480\times5}{7}$
4^2	5^2	$-\frac{480\times7}{11}$	$\frac{480\times28}{11}$
6^2	5^2	$\frac{2800\times24}{11\times13}$	$\frac{2800\times168}{11\times13}$
6^2	7^2	$-\frac{256\times63}{13}$	$\frac{256\times63\times6}{13}$
3^3	4^3	-800	15×160
5^3	4^3	$\frac{32\times32\times35}{11}$	$\frac{32\times32\times35\times6}{11}$
5^3	6^3	$-\frac{840\times3024}{11\times13}$	$\frac{315\times40\times1008}{11\times13}$
7^3	6^3	$\frac{120\times72\times56}{13}$	$\frac{960\times72\times56}{13}$
7^3	8^3	$-\frac{220\times126\times72}{17}$	$\frac{220\times72\times14\times63}{17}$

$n_\alpha^{\ m}$	$n_\beta^{\ m}$	I_1	I_2
2^0	1^0	$\frac{4}{15}$	$\frac{4}{5}$
2^0	3^0	$-\frac{24}{35}$	$\frac{48}{35}$
4^0	3^0	$\frac{8}{21}$	$\frac{40}{21}$
4^0	5^0	$-\frac{20}{33}$	$\frac{80}{33}$
6^0	5^0	$\frac{60}{11\times13}$	$\frac{15\times28}{11\times13}$
6^0	7^0	$-\frac{16\times7}{13\times15}$	$\frac{96\times7}{13\times15}$
2^1	1^1	$\frac{4}{5}$	$\frac{12}{5}$
2^1	3^1	$-\frac{192}{35}$	$\frac{384}{35}$
4^1	3^1	$\frac{40}{7}$	$\frac{200}{7}$
4^1	5^1	$-\frac{160}{11}$	$\frac{640}{11}$
6^1	5^1	$\frac{105\times20}{18\times11}$	$\frac{735\times20}{18\times11}$
6^1	7^1	$-\frac{16\times8\times14}{13\times5}$	$\frac{96\times8\times14}{13\times5}$
3^2	2^2	$\frac{96}{7}$	$\frac{384}{7}$
3^2	4^2	$-\frac{800}{7}$	$\frac{2400}{7}$
5^2	4^2	$\frac{32\times70}{11}$	$\frac{480\times28}{11}$
5^2	6^2	$-\frac{735\times128}{11\times13}$	$\frac{2800\times168}{11\times13}$
7^2	6^2	$\frac{192\times63}{13}$	$\frac{256\times63\times6}{13}$
7^2	8^2	$-\frac{81\times64\times7}{17}$	$\frac{567\times64\times7}{17}$

Table II λ in 10^4.

Ω^2	(a.1)	(a.2)	(b.1)	(c.1)
0	1.060	1.06		
10^2	1.079	1.08	1.078	1.078
10^3	1.247	1.27	1.236	1.235
10^4	2.930	3.03	2.359	2.311

Table III λ in 10^4

Ω^2	$u_{1.0}$(I)	$u_{1.0}$(II)	$u_{1.1}$(I)	$u_{1.1}$(II)	$u_{2.2}$(I)	$u_{2.2}$(II)	$u_{3.3}$(I
0	0.811	1.06	0.811	1.06	1.06	1.53	1.53
10^2	0.862	1.11	0.849	1.10	1.08	1.55	1.54
10^3	1.225	1.55	1.13	1.48	1.235	1.79	1.64
10^4	2.68	3.70	2.48	3.47	2.31	3.52	2.44

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